ON STRONGLY PRIME AND ONE-SIDED STRONGLY PRIME SUBMODULES

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Outline

Introduction and Preliminaries

Before stating our new results, we would like to list some basic properties from [5].

Definition 1.1

A proper ideal P in a ring R is called a *prime ideal* of R if for any ideals I, J of R with $IJ \subset P$, then either $I \subset P$ or $J \subset P$. An ideal I of a ring R is called *strongly prime* if for any $a, b \in R$ with $ab \in I$, then either $a \in I$ or $b \in I$. A ring R is called a *prime ring* if 0 is a prime ideal.

Proposition 1.2

For a proper ideal P in a ring R, the following conditions are equivalent:

(1) P is a prime ideal;

(2) If I and J are any ideals of R properly containing P, then $IJ \not\subseteq P$;

(3) R/P is a prime ring;

(4) If I and J are any right ideals of R such that $IJ \subset P$, then either $I \subset P$ or $J \subset P$;

(5) If I and J are any left ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$;

(6) If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$;

Introduction and Preliminaries

In modifying the structure of prime ideals and prime rings, many authors transfered these notions to modules. There are many ways to generalize these notions and it is an effective way to study structures of modules. For examples, Andrunakievich and Dauns [1961], Beachy and Blair [1975], Dauns [1978], Bican et.al. [1980], Wisbauer [1983], C. P. Lu [1984], Behboodi and Koohy [2004] gave some definitions of prime submodules. However, from these definitions, we could not find any properties which are similar to the properties of prime ideals. In 2008, N. V. Sanh [7] proposed a new definition of prime submodules. By this definition, we found many beautiful properties of prime submodules that are similar to prime ideals. We constructed some new notions such as nilpotent submodules, nil submodules, a prime radical, a nil radical and a Levitzki radical of a right module M over an arbitrary associative ring R and described all properties of them as generalizations of nilpotent ideals, nil ideals, a prime radical, a nil radical and a Levitzki radical of rings.

Introduction and Preliminaries

Throughout this report, all rings are associative rings with identity and all modules are unitary right R- modules. Let R be a ring and M, a right R-module. Denote $S = End_R(M)$, the endomorphism ring of the module M. A submodule X of M is called a *fully invariant* submodule if $f(X) \subset X$, for any $f \in S$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R.

A right R-module M is called a *self-generator* if it generates all its submodules.

Definition 1.3

A fully invariant submodule X of M is called a *prime submodule* of M if for any ideal I of $S = End_R(M)$, and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. A proper ideal P in a ring R is called a *prime ideal* of R if for any ideals I, J of R with $IJ \subset P$, then either $I \subset P$ or $J \subset P$.

Theorem 1.4

Let X be a proper fully invariant submodule of M. Then the following conditions are equivalent:

(1) X is a prime submodule of M;

(2) For any right ideal I of S, any submodule U of M, if $I(U) \subset X$, then either $I(U) \subset X$ or $U \subset X$;

(3) For any $\varphi \in S$ and fully invariant submodule U of M, if $\varphi(U) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$;

(4) For any left ideal I of S and subset A of M, if $IS(A) \subset X$, then either $I(M) \subset X$ or $A \subset X$;

(5) For any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$, then either $\varphi(M) \in X$ or $m \in X$.

Corollary 1.5

For a proper ideal P in a ring R, the following conditions are equivalent:

(1) P is a prime ideal;

(2) If I and J are any ideals of R properly containing P, then $IJ \nsubseteq P$;

(3) R/P is a prime ring;

(4) If I and J are any right ideals of R such that $IJ \subset P$, then either $I \subset P$ or $J \subset P$;

(5) If I and J are any left ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$;

(6) If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$;

Introduction and Preliminaries

Lemma 1.6

Let M be a right R-module and $S = End(M_R)$. Suppose that X is a fully invariant submodule of M. Then the set $I_X = \{f \in S | f(M) \subset X\}$ is a two-sided ideal of S.

Theorem 1.7

Let M be a right R- module, $S = End(M_R)$ and X a fully invariant submodule of M. If X is a prime submodule of M, then I_X is a prime ideal of S. Conversely, if M is a self-generator and if I_X is a prime ideal of S, then X is a prime submodule of M.

Strongly prime submodules

Definition 2.1

A fully invariant submodule U of M is called *strongly prime* if for any $f \in S$, any $m \in M$, $f(m) \in U$, then either $f(M) \subset U$ or $m \in U$. Especially, an ideal I of a ring R is *strongly prime* if for any $a, b \in R, ab \in I$, then either $a \in I$ or $b \in I$.

Remark 2.2

(1) Every strongly prime submodule is prime.

(2) Every prime submodule in a duo module is a strongly prime submodule.

Strongly prime submodules

An R-module M is called *strongly prime* if 0 is a strongly prime submodule of M.

A ring R is called *strongly prime* if 0 is a strongly prime ideal of R.

Proposition 2.3

Let M be a right R-module which is a quasi-projective module. Then the following are equivalent:

(1) X is a strongly prime submodule of M,

(2) M/X is a strongly prime module.

Corollary 2.4

Let I be an ideal of the ring R. Then I is a strongly prime ideal if and only if R/I is a strongly prime ring.

Lemma 2.5

Let M, N be right R-modules and $f : M \longrightarrow N$ be an epimorphism. Suppose that $Kerf$ is a fully invariant submodule of M. Then,

(1) For any $\varphi \in S$, there exists $\phi \in End(N)$ such that $\phi f = f\varphi$.

(2) If V is a fully invariant submodule of N, then $U = f^{-1}(V)$ is a fully invariant submodule of M.

Lemma 2.6

Let M be a quasi-projective module and P, a strongly prime submodule of M. If $A \subset P$ is a fully invariant submodule of M, then P/A is a strongly prime submodule of M/A.

Strongly prime submodules

Proposition 2.7

Let M be a quasi-projective module and $f : M \longrightarrow N$ be an epimorphism such that $Kerf$ is a fully invariant submodule of M. Then,

(1) If Y is a strongly prime submodule of N, then $X = f^{-1}(Y)$ is a strongly prime submodule of M.

(2) If X is a strongly prime submodule of M, then $f(X)$ is a strongly prime submodule of N.

Strongly prime submodules

Definition 2.8

A submodule X of a right R-module M is said to have *"insertion factor property"* (briefly, an IFP-submodule) if for any endomorphism φ of M and any element $m \in M$, if $\varphi(m) \in X$, then $\varphi S m \subset X$. A right ideal I is an IFP- right ideal if it is an IFP submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right R-module M is called an IFP-module if 0 is an IFP-submodule of M. A ring is IFP if 0 is an IFP-ideal.

We have the relationship between an IFP module M and the ring S, its endomorphism ring by the following theorem.

Theorem 2.9

If X is an IFP submodule of M, then I_X is is an IFP right ideal of S. The converse is true if M is a self-generator.

Strongly prime submodules

Theorem 2.10

Let M be an R-module. A submodule X of M is a strongly prime submodule if and only if it is prime and IFP. An ideal I of a ring R is a strongly prime ideal if and only if it is prime and IFP.

Strongly prime submodules

Theorem 2.11

Let M be a right R-module. If X is a strongly prime submodule of M, then I_X is a strongly prime ideal of S. Conversely, if M is a self-generator and I_X is a strongly prime ideal of S, then X is a strongly prime submodule.

Definition 3.1

A submodule U of M is called *one-sided strongly prime* if for any $f \in S$ and $m \in M$ such that $f(U) \subset U$ and $f(m) \in U$, then either $f(M) \subset U$ or $m \in U$. In particular, a right ideal $P \subset R$ is an *one-sided strongly prime right ideal* if for any $a, b \in R$ such that $aP \subset P$, $ab \in P$, then either $a \in P$ or $b \in P$.

Example 3.2

(1) Let $\mathbb{M}_3(k)$ be a matrix ring and k be a division ring. Let R be the following subring of $\mathbb{M}_3(k)$:

$$R := \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

Let $P \subset R$ be the right ideal of R of the form $P := \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$. Then P is an one-sided strongly prime right ideal of R.

Example 3.2

(2) Every maximal submodule is an one-sided strongly prime submodule. In particular, every maximal right ideal of a ring R is an one-sided strongly prime right ideal.

(3) Every prime submodule in a duo module is a strongly prime submodule. Hence, every prime ideal in a right duo ring is a strongly prime ideal.

Theorem 3.3

Let M be a right R-module which is a self-generator. If X is an one-sided strongly prime submodule of M, then I_X is an one-sided strongly prime right ideal of S. Conversely, if I_X is an one-sided strongly prime right ideal of S, then X is an one-sided strongly prime submodule of M.

On one-sided strongly prime submodules

It was shown in [9, Theorem 2.1] that if M is a quasi-projective, finitely generated right R-module and M is a Noetherian module, then S is a right Noetherian ring. For the converse, we need M to be a self-generator as in the next theorem.

Theorem 3.4

Let M be a right R-module which is a self-generator. If S is a right Noetherian ring, then M is a Noetherian module.

On one-sided strongly prime submodules

In 1950, I. S. Cohen showed that if every prime ideal in a commutative ring with identity is finitely generated, then R is Noetherian. Many authors modified Cohen theorem for noncommutative rings. In 1975, R. Chandran [3] proved that it is true for the class of duo rings. In 2011, M. L. Reyes [6] introduced the notion of completely prime right ideals. A right ideal $P \subset R$ is a completely prime right ideal if for any $a, b \in R$ such that $aP \subset P$ and $ab \in P$, then either $a \in P$ or $b \in P$. By this definition, M. L. Reyes proved that a ring R is right Noetherian if and only if all of its completely prime right ideals are finitely generated. We present it here for convenience.

Theorem 3.5

A ring R is right Noetherian if and only if all of its completely prime right ideals are finitely generated.

Theorem 3.6

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If every one-sided strongly prime submodule of M is finitely generated, then M is a Noetherian right R-module.

Recall that a right R- module M is called a *duo module* if every submodule of M is a fully invariant submodule of M. A ring is called a right duo ring if every ideal is a two-sided ideal.

It is easy to see that a fully invariant one-sided strongly prime submodule of M is a strongly prime submodule of M. Thus, if M is a duo module, then every one-sided strongly prime submodule of M is also a strongly prime submodule of M. Using this result, we have the following corollary.

Corollary 3.7

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a duo module and every strongly prime submodule of M is finitely generated, then S is a right Noetherian ring, and hence M is Noetherian.

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